IA Groups – Example Sheet 2

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Questions marked * are more challenging.

1. Show that if two elements of a group are conjugate, then they have the same order.

2. Let H be a subgroup of a group G. Show that there is a bijection between the set of left cosets of H in G and the set of right cosets of H in G.

3. Show that if a group G contains an element of order 6 and an element of order 10, then G has order at least 30.

4. For H a subgroup of a finite group G, and K a subgroup of H, show that

$$|G : K| = |G : H| \cdot |H : K| .$$

   * What happens when G is infinite?

5. Suppose that a group G acts on a set X.

   (a) Show that
   $$\mathrm{Stab}_G(hx) = h\,\mathrm{Stab}_G(x)\,h^{-1}$$
   for any $x \in X$ and $h \in G$.

   (b) For any $g \in G$, let $\mathrm{Fix}(g) = \{y \in X \mid gy = y\}$ be the set of points fixed by g. Show that
   $$\mathrm{Fix}(hgh^{-1}) = h\,\mathrm{Fix}(g)$$
   for any $h \in G$.

6. Show that D_{2n} has one conjugacy class of reflections if n is odd and two conjugacy classes of reflections if n is even. Draw a picture to illustrate your answer.

7. Let G be the group of all isometries of a cube in $\mathbb{R}^3$. Show that G acts on the set of 4 lines that join diagonally opposite pairs of vertices. * Show that if ℓ is one of these lines then $\mathrm{Stab}_G(\ell) \cong D_{12}$.

8. Let G be a finite abelian group acting faithfully on a set X. Show that if the action is transitive then $|G| = |X|$.

9. Let G be a finite group and let $\mathrm{Sub}(G)$ be the set of all its subgroups. Show that

$$g(H) = gHg^{-1} = \{ghg^{-1} \mid h \in H\}$$

   defines an action of G on $\mathrm{Sub}(G)$. Show that, for any $H \in \mathrm{Sub}(G)$, the size of the orbit of H under this action is at most $|G : H|$. Deduce that if $H \neq G$ then G is not the union of all conjugates of H.

10. Let G be a finite group acting on a set X. By counting the set $\{(g, x) \in G \times X \mid g(x) = x\}$ in two ways, show that the number of orbits of the action is equal to

$$\frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)| .$$

   [*This famous result is called 'Burnside's lemma'.*] Deduce that if G acts transitively and $|X| > 1$, then there is some $g \in G$ with no fixed point.

11. Express the Möbius transformation $f(z) = \frac{2z+3}{z-4}$ as the composition of transformations of the form

$$\alpha_a : z \mapsto az\,,\ \beta_b : z \mapsto z + b\,,\ \gamma : z \mapsto 1/z\,.$$

Hence show that f sends the circle described by $|z-2i| = 2$ to the circle described by $|8z+(6+11i)| = 11$.

12. Consider the Möbius transformations $f(z) = e^{2\pi i/n}z$ and $g(z) = 1/z$. Show that the subgroup G of the Möbius group $\mathcal{M}$ generated by f and g is isomorphic to D_{2n}.

13. Let G be the subgroup of Möbius transformations that send the set $\{0, 1, \infty\}$ to itself. List the elements of G. Identify G. Identify the group H of Möbius transformations that send the set $\{0, 2, \infty\}$ to itself by relating H to G.

[*Here, 'identify' means 'find a standard group that it is isomorphic to'.*]

14. Prove or disprove each of the following statements:

    (i) The Möbius group is generated by Möbius transformations of the form $\alpha_a : z \mapsto az$ and $\beta_b : z \mapsto z + b$.

    (ii) The Möbius group is generated by Möbius transformations of the form $\alpha_a : z \mapsto az$ and $\gamma : z \mapsto 1/z$.

    (iii) The Möbius group is generated by Möbius transformations of the form $\beta_b : z \mapsto z + b$ and $\gamma : z \mapsto 1/z$.

15. Determine under what conditions on $\lambda, \mu \in \mathbb{C} \setminus \{0\}$ the Möbius transformations $f(z) = \lambda z$ and $g(z) = \mu z$ are conjugate in $\mathcal{M}$.

16. What is the order of the Möbius transformation $f(z) = iz$? What are its fixed points? Construct a Möbius transformation of order 4 that fixes 1 and -1.